C_L^* = concentration in liquid in equilibrium with gas, mole/cm^3
$\overline{C}$ = arithmetic average concentration, mole/cm^3
C_x = concentration of active sites on catalysts, number of sites/(g of catalyst)
D = molecular diffusivity in liquid, cm^2/s
d_p = equivalent spherical, particle diameter, cm
E = axial dispersion coefficient, cm^2/s
E' = activation energy, Equation (3), kcal/mole
E'' = constant in Equation (3), kcal/mole
F_g = volumetric flow rate of gas at 25°C, 1 atm, cm^3/s
F_L = volumetric flow rate of liquid at reactor temperature and pressure, except where noted, cm^3/s
G_a = Gallileo number, $d_p{}^3\, g\, \rho^2/\mu^2$
G_L = superficial mass velocity of liquid, g/(cm^2 s)
g = acceleration of gravity, cm/s^2
H' = Henry's law constant, atm; defined as $P_i = H'x_i$ where x_i = liquid phase mole fraction of component i at partial pressure P_i
H_{O_2} = Henry's law constant (cm^3 of liquid at temperature and pressure)/(cm^3 of gas at 25°C and 1 atm), $C_g = HC_L^*$
H_d = dynamic holdup, (cm^3 of liquid)/(cm^3 of empty reactor)
K' = constant in Equation (2), (cm)$^{4.5}$/[(mole)$^{0.5}$ (g s)]
K'' = constant in Equation (2), cm^3/mole
K_{HA} = adsorption equilibrium constant for acetic acid
K_{O_2} = adsorption equilibrium constant for oxygen
k_L = mass transfer coefficient from gas-liquid interface to liquid, cm/s
k_s = mass transfer coefficient from liquid to particle surface, cm/s
n_L = parameter in Equation (15)
n_s = parameter in Equation (16)
Pe = axial Peclet number, $d_p\, u_L/E$
R = reaction rate, mole/(g s)
Re = Reynolds number, $d_p\, G_L/\mu$
S = cross-sectional area of reactor, cm^2
T = temperature, °K
t = temperature, °C
u_L = superficial velocity of liquid, cm/s
W = mass of catalyst, g
X = active catalyst site
X_{HA} = conversion of acetic acid in liquid at the exit of the reactor
$Y_{g,O_2} = C_{g,O_2}/(C_{g,O_2})_f$
$Y_{L,O_2} = C_{L,O_2}/(C^*_{L,O_2})_f$
$Y_{s,O_2} = C_{s,O_2}/(C^*_{L,O_2})_f$
$Y_{L,HA} = C_{L,HA}/(C_{L,HA})_f$
$Y_{s,HA} = C_{s,HA}/(C_{s,HA})_f$
z = axial coordinate in reactor, cm
z_B = total length of catalyst bed, cm
α_L = parameter in Equation (15), (cm)$^{n_L-2}$
α_s = parameter in Equation (16), (cm)$^{n_s-2}$
ρ_B = bulk density of catalyst bed, g/cm^3 of reactor filled with catalyst particles
μ = fluid viscosity, g/(cm s)

Subscripts

HA = acetic acid
e = exit
f = feed
g = gas phase
i = component i
L = liquid phase
s = particle surface

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Convective Instabilities in Porous Media with Through Flow

GEORGE M. HOMSY
Department of Chemical Engineering
Stanford University
Stanford, California 94305

and

ALBERT E. SHERWOOD
Lawrence Livermore Laboratory,
University of California
Livermore, California 94550

The stability of a thermally stratified, saturated porous media through which mass is being ejected is considered theoretically. The stability parameter is a flow modified D'Arcy-Rayleigh number and is a function of a single scalar variable, the dimensionless through-flow strength. Results of both linear and energy theory are given, and it is seen that the fluid can lose stability by either a buoyantly driven mode or by a continuous analogue of the Saffman-Taylor mode.

SCOPE

Many operations and processes involving the nonisothermal flow of fluid through saturated porous media commonly occur in geophysics, packed-bed processing, in situ coal gasification, and other areas. The question of the occurrence of buoyantly driven instabilities naturally arises for these processes. The purpose of this paper is to provide as comprehensive a stability theory as possible for the important case of net fluid discharge through a medium heated uniformly from below. Both linear limits, giving sufficient conditions for instability, and energy limits, giving sufficient conditions for stability, are developed in this study. The linear instability theory has been previously studied only in the limits of zero or weak through flow by Lapwood (1948) and Sutton (1970) and very strong through flow by Wooding (1960). The energy limit has been studied only for no through flow by Westbrook (1969). The results appear in terms of a flow modified D'Arcy-Rayleigh number as a function of through-flow strength. Thus, given a temperature contrast and discharge velocity, the theory makes definitive statements with regard to stability or instability.

CONCLUSIONS AND SIGNIFICANCE

The model employed here is that of a fluid in a saturated porous medium heated uniformly below with a superimposed flow parallel to the gravity vector. The model bears relevance to a variety of in situ processing schemes, as well as to nonisothermal packed-bed operations. It is important to be able to make statements regarding the stability/instability of the flow. It is shown, in agreement with Wooding (1960), that the stability parameter is the D'Arcy-Rayleigh number, which consists of two terms [see definitions following Equation (9)]. The first term is a familiar ratio of buoyant forces to dissipative forces, and the second arises from the temperature variation of viscosity, coupled with the superimposed axial velocity. Thus, for a sufficiently large D'Arcy-Rayleigh number, a layer of fluid loses its stability by two mechanisms, the first of which is a buoyant mode, and the second is loosely analogous to the Saffman-Taylor (1958) mode.

The main results of this study are shown in Figure 1, where both the linear and energy limits are given as a function of dimensionless through-flow strength. Below the energy limit is certain stability; above the linear limit the theory guarantees instability. The region in between is stable to small disturbances; it may be unstable to disturbances with sufficiently large amplitude. At low discharge rates, the two limits are coincident; at higher rates, they differ but remain close, after a suitable rescaling [Equation (24)]. Asymptotic representations for very strong discharge rates are determined, and it is shown that stability/instability criteria become independent of layer depth and thermal diffusivity.

In the limit for high discharge, a second group, designated as the D'Arcy buoyancy number, is the controlling stability parameter. It is shown that to the extent that the Saffman-Taylor mode is negligible, either upflow or downflow exerts a stabilizing effect.

The in situ processing of energy resources such as coal, oil shale, or geothermal energy, often involves the nonisothermal flow of fluids through porous media. In some cases, vertical processing of the resource has been suggested (Higgins, 1972), and the question arises as to how important buoyantly driven instabilities might become in such systems. The same considerations apply to processing in packed-bed reactors. Taken in a wider scope, the problem of buoyantly driven fluid motions in geological formations impacts many areas of geophysics (Elder, 1967).

The mathematical structure of the equations governing natural convection in saturated porous media is itself of interest. The D'Arcy-Boussinesq equations are of lower order than those normally studied, which makes detailed and physically complex problems mathematically tractable. Thus, considerable analytical and numerical progress can be made toward the theoretical understanding of such flows.

The present study is intended to give a fairly complete account of the stability of an unstable stratified fluid in a saturated porous medium, including the effects of fluid through flow (so-called mass-discharge). This through flow is an integral feature of in situ processing, and it is of interest to assess its effect on the stability limits.

The stability theory in the case of an initially stagnant fluid is well understood. The instability parameter of interest in this case is the D'Arcy-Rayleigh number

$$R^2 = \frac{g\alpha\Delta Tkl}{\nu\kappa} \tag{1}$$

For $R^2 > 4\pi^2$, the linear theory of Horton and Rodgers (1945) and Lapwood (1948) indicates that the layer is convectively unstable to disturbances of small amplitude. The method of energy has also been applied to this problem, initially by Westbrook (1969). Certain errors of no great consequence have been corrected by Wankat and Schowalter (1970). The energy limit that gives sufficient

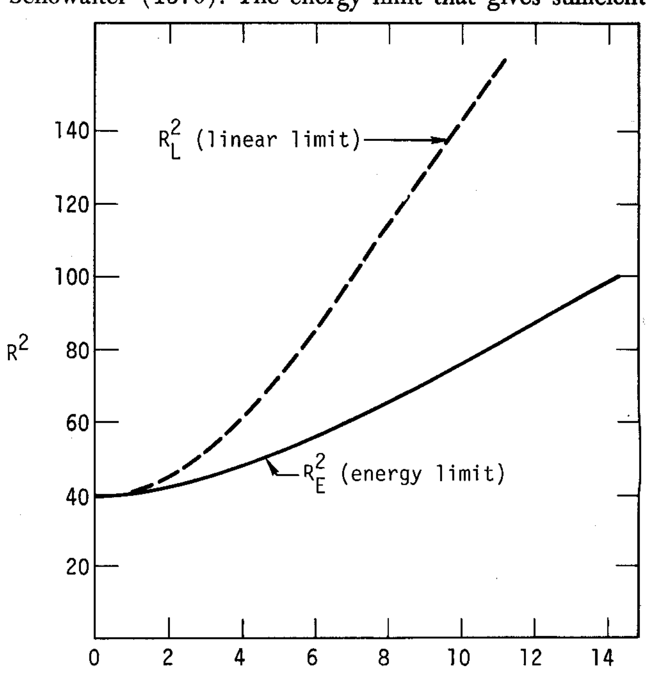

Fig. 1. The stability limits as a function of dimensionless discharge rate.

conditions for stability is, in this case, identical to the linear limit. Thus, for $R^2 < 4\pi^2$, the layer is stable for disturbances of arbitrary amplitude. The experimental results of Elder (1967) and Combarnous and LeFur (1969) have confirmed these predictions. The finite amplitude problem has recently been studied by Straus (1974). Thus, a fairly complete theoretical and experimental picture of the problem exists for the case of no mass discharge.

To our knowledge, only two previous papers have addressed the problem of the effect of vertical mass discharge on the stability limit. Wooding (1960) has treated the case in which the base state temperature field is dominated by the convective effects of the mean flow. In this limit, the layer appears semiinfinite, and the onset of instability is measured by a Rayleigh number in which the depth l in (1) is replaced by the thermal depth κ/w_o, and the definition of the Rayleigh number is modified. Here, w_o is the magnitude of the through-flow velocity. For a special choice of boundary conditions, Wooding finds a flow modified Rayleigh number for onset. His work is discussed in further detail later. In a short note, Sutton (1970) has presented linear limits valid for very small through flow.

The purpose of the present paper is to present as detailed a stability theory as possible for both strong and weak mass discharge. A detailed discussion of the implications of these results vis-à-vis in situ coal gasification is given elsewhere (Sherwood and Homsy, 1975).

PRELIMINARIES

Basic Equations

We take as the basic equations the following:

$$\epsilon \frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{v}) = 0 \quad (2a)$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\epsilon} \mathbf{v} \bullet \nabla \mathbf{v} \right) = - \nabla p + \rho \mathbf{g} - \frac{\mu}{k} \mathbf{v} \quad (2b)$$

$$A \frac{\partial T}{\partial t} + \mathbf{v} \bullet \nabla T = \kappa \nabla^2 T \quad (2c)$$

Here, $A = \rho_m c_{pm} / \rho_f c_{pf}$, the ratio of the volumetric heat capacity of fluid/solid mixture to fluid, and $\kappa = k_m / \rho_f c_{pf}$, an effective thermal diffusivity. There is still controversy surrounding the form of the inertial term in the momentum equation. The $\mathbf{v} \bullet \nabla \mathbf{v}$ term is associated with inertia on the continuum length scale and arises from a formal volume averaging of the point field equations (Drew and Segel, 1971). Other workers, drawing from experience in packed beds, have suggested an inertial term of the form $\mathbf{v} | \mathbf{v} |$ (Irmay, 1958). The inertial term is associated with microscale inertial effects and is negligible if a Reynolds number based upon interstitial velocity and pore length or grain size is small. We make such an assumption here and rule out any non-D'Arcy effects.

The Base State

We seek a solution to the basic equations subject to the boundary conditions of constant flow in the vertical direction and constant fixed temperatures T_o and T_1 at $z = 0$ and $z = l$, respectively. We assume the following linear variation of fluid properties with temperature

$$\rho = \rho_o [1 - \alpha(T - T_o)] \quad (3a)$$

$$\mu = \mu_o [1 - \beta(T - T_o)] \quad (3b)$$

where α is the coefficient of thermal expansion

$$\beta \equiv \frac{-1}{\mu_o} \left(\frac{\partial \mu}{\partial T} \right)_{T_o},$$

and $\{\alpha, \beta\}$ are considered to be small parameters. We set

$$\mathbf{v} = w_o \mathbf{k} \quad (4)$$

where w_o is the magnitude of the through flow. The steady equations then become, to lowest order in α

$$- \nabla p + \rho \mathbf{g} - \frac{\mu}{k} w_o \mathbf{k} = 0 \quad (5a)$$

$$w_o \frac{dT}{dz} = \kappa \frac{d^2 T}{dz^2} \quad (5b)$$

where ρ, μ in Equation (5a) are given by Equation (3). The continuity Equation (2a) is satisfied identically to the lowest order in α. When constant mass flux is specified rather than constant velocity, the development is slightly different (Wooding, 1960). The solution to the boundary value problem for the temperature is found simply as

$$\overline{\theta} \equiv T - T_o = \frac{(T_1 - T_o)(1 - e^{w_o z/\kappa})}{1 - e^{w_o l/\kappa}} \quad (6)$$

The pressure distribution may be found as an integral of Equation (5a), but it is of no consequence in what follows. Thus, the base state consists of Equations (4) and (6) plus the pressure p.

The Disturbance Equations and the Boussinesq Approximation

We wish to obtain equations for the disturbance quantities. It is convenient to define the following quantities:

$$T - T_o = \overline{\theta}(z) + \theta'(x, y, z, t)$$

$$\mathbf{v} = w_o \mathbf{k} + \mathbf{v}'$$

$$p = \overline{p} + p'$$

Then, by Equation (3)

$$\rho = \rho_o (1 - \alpha [\overline{\theta} + \theta'])$$

$$\mu = \mu_o (1 - \beta [\overline{\theta} + \theta'])$$

The full field equations for the disturbance quantities are

$$\rho_o (\nabla \bullet \mathbf{v}') + \epsilon \frac{\partial \rho'}{\partial t} - \rho_o \alpha \nabla \bullet (\mathbf{v} [\overline{\theta} + \theta']) = 0 \quad (7a)$$

$$\rho \left[\frac{\partial \mathbf{v}'}{\partial t} + \frac{1}{\epsilon} \mathbf{v}' \bullet \nabla \mathbf{v}' + \frac{1}{\epsilon} w_o \frac{\partial \mathbf{v}'}{\partial z} \right] = - \nabla p'$$

$$+ \rho_o g \alpha \theta' \mathbf{k} - \frac{\mu_o}{k} [(1 - \beta \overline{\theta}) v' - \beta \theta' w_o \mathbf{k} - \beta \theta' \mathbf{v}'] \quad (7b)$$

$$A(T_o) \frac{\partial \theta'}{\partial t} + w_o \frac{\partial \theta'}{\partial z} + (\mathbf{k} \bullet \mathbf{v}') \frac{\partial \overline{\theta}}{\partial z}$$

$$+ v' \bullet \nabla \theta' = \kappa \nabla^2 \theta' \quad (7c)$$

The Boussinesq approximation, then, amounts to assuming the formal series solution for disturbance quantities

$$\begin{bmatrix} \rho' \\ v' \\ \theta' \end{bmatrix} = \sum_{m,n} \begin{bmatrix} \rho'_{mn} \\ v'_{mn} \\ \theta'_{mn} \end{bmatrix} \beta^m \alpha^n$$

and truncating at lowest order. In some applications, non-Boussinesq effects may be important, and the results obtained on the basis of the Boussinesq equations may be only qualitatively correct. The resulting D'Arcy-Boussinesq equations are

$$\nabla \cdot \mathbf{v}' = 0 \quad (8a)$$

$$\rho_o \left[\frac{\partial \mathbf{v}'}{\partial t} + \frac{1}{\epsilon} \mathbf{v}' \bullet \nabla \mathbf{v}' + \frac{1}{\epsilon} w_o \frac{\partial \mathbf{v}'}{\partial z} \right] = - \nabla p'$$

$$+ \theta'\mathbf{k}\left[\rho_o g\alpha + \frac{\mu_o}{k} w_o\beta\right] - \frac{\mu_o}{k}\mathbf{v}' \quad (8b)$$

$$A(T_o)\frac{\partial\theta'}{\partial t} + w_o\frac{\partial\theta'}{\partial z} + (\mathbf{k}\bullet\mathbf{v}')\frac{\partial\theta'}{\partial z} + \mathbf{v}'\bullet\nabla\theta' = \kappa\nabla^2\theta' \quad (8c)$$

It is convenient to make the variables dimensionless with respect to the scalings

$$\{r, v', \theta', t, p\} = \left\{ l, \kappa/l, (T_o - T_1), l^2/\kappa, \frac{\kappa\mu_o}{k} \right\}$$

When this is done, we arrive at the following dimensionless equations for the disturbance quantities (dropping the primes):

$$\nabla\bullet\mathbf{v} = 0 \quad (9a)$$

$$\frac{1}{\sigma}\left[\frac{\partial\mathbf{v}}{\partial t} + \frac{1}{\epsilon}\mathbf{v}\bullet\nabla\mathbf{v}\right] + \left(\frac{kw_o}{\epsilon l\nu_o}\right)\frac{\partial\mathbf{v}}{\partial z} = - \nabla p + R^2\theta\mathbf{k} - \mathbf{v} \quad (9b)$$

$$A(T_o)\frac{\partial\theta}{\partial t} + \gamma\frac{\partial\theta}{\partial z} + (\mathbf{k}\bullet\mathbf{v})\frac{\partial\overline{\theta}}{\partial z} + \mathbf{v}\bullet\nabla\theta = \nabla^2\theta \quad (9c)$$

The boundary conditions are $\theta = \mathbf{v} = 0$, $z = 0, 1$.

The four dimensionless parameters of the problem are:

1. A flow modified D'Arcy-Rayleigh number

$$R^2 = \frac{g\alpha\Delta Tkl}{\nu\kappa} + \gamma\beta\Delta T$$

2. A dimensionless flow strength

$$\gamma = \frac{w_o l}{\kappa}$$

3. The D'Arcy-Prandtl number

$$\sigma = \frac{\nu}{\kappa}\,\frac{l^2}{k}$$

4. A D'Arcy-Reynolds number

$$\frac{kw_o}{\nu l}$$

For the case of constant mass flux, a similar development leads to the definition (Sherwood and Homsy, 1975)

$$R^2 = \frac{g\alpha\Delta Tkl}{\nu\kappa} + \gamma(\beta - \alpha)\Delta T$$

The numerical results below are equally valid for this case; the interpretation changes only slightly.

We note here that the dimensionless base state gradient $\partial\overline{\theta}/\partial z$ enters into the energy equation (9c). From Equation (6) we find

$$\frac{\partial\overline{\theta}}{\partial z} = \frac{\gamma e^{\gamma z}}{1 - e^{\gamma}}$$

which for small γ approaches the conductive limit $\partial\overline{\theta}/\partial z \rightarrow -1$ and for large γ approaches that given by Wooding (1960). The parameter γ may be interpreted as the ratio of the fluid depth l to the thermal depth κ/w_o. For large $|\gamma|$, the gradient is confined to thermal boundary layers of dimensionless thickness γ^{-1} near the top or bottom of the layer for positive or negative vertical velocity, respectively.

The problem is thus to predict the critical flow modified D'Arcy-Rayleigh number for which the layer is unstable or stable as a function of the three other parameters of the problem. The linear instability problem is presented next, followed by the energy stability problem.

LINEAR INSTABILITY THEORY

Predictions of sufficient conditions for instability can be made by considering the linearization of Equation (9) for disturbances of infinitesimal amplitude. Neglecting those terms which are nonlinear in the disturbance quantities, we arrive at

$$\nabla\bullet\mathbf{v} = 0 \quad (10a)$$

$$\frac{1}{\sigma}\frac{\partial\mathbf{v}}{\partial t} + \left(\frac{kw_o}{\epsilon l\nu_o}\right)\frac{\partial\mathbf{v}}{\partial z} = - \nabla p + R^2\theta\mathbf{k} - \mathbf{v} \quad (10b)$$

$$A(T_o)\frac{\partial\theta}{\partial t} + \gamma\frac{\partial\theta}{\partial z} + w\frac{\partial\overline{\theta}}{\partial z} = \nabla^2\theta \quad (10c)$$

where $w = \mathbf{k}\bullet\mathbf{v}$ is the disturbance vertical velocity.

It is possible to prove an exchange-of-stabilities argument in the limit of zero ($\gamma \rightarrow 0$) or strong ($\gamma >> 1$) through flow. The proof in the former case is straightforward; for the latter see Wooding (1960). Thus, onset must be by the neutral mode $\partial/\partial t = 0$. The marginal stability curve is thus obtained by the solution of the steady version of (10). Furthermore, the D'Arcy-Reynolds number is extremely small (on the order of 10^{-6}) for typical values of the variables, which implies that the second term on the left-hand side of Equation (10b) is negligible under most circumstances. The net effect of these two arguments is equivalent to the a priori neglect of inertia and acceleration effects in the momentum equation (Wooding, 1960).

Neglect of the inertial term has unknown mathematical consequences and should perhaps not be dismissed so lightly. An obvious consequence of its omission is to lower the order of the equations (9b), and hence its neglect, if indeed it belongs in the field equations to begin with, is not uniformly valid and must be justified in the sense of singular perturbations. Beck (1972) has recently raised this question and argues against the inclusion of $\mathbf{v}\bullet\nabla\mathbf{v}$ on the pragmatic grounds that boundary conditions for the inertial equations are at present unknown. What is more likely, however, is that (2b) are themselves incomplete, and that a term of the form $\nabla^2\mathbf{v}$, with a small coefficient, must be present, Batchelor (1974). The full equations then support the usual no-slip conditions in singular regions near boundaries and reduce to (11b) in regions far removed from solid walls. Pursuit of these questions, however, is beyond the scope of this paper.

We are thus left with the marginal stability equations

$$\nabla\bullet\mathbf{v} = 0 \quad (11a)$$

$$0 = - \nabla p + R^2\theta\mathbf{k} - \mathbf{v} \quad (11b)$$

$$\gamma\frac{\partial\theta}{\partial z} + w\frac{\partial\overline{\theta}}{\partial z} = \nabla^2\theta \quad (11c)$$

where

$$\mathbf{v} = 0 \quad \text{and} \quad \theta = 0, \quad \text{at} \quad z = 0, 1$$

The solenoidal field $\mathbf{v}$ is completely determined by specification of w and the vertical component of vorticity ζ. By taking the vertical component of the curl of Equation (11b), it is possible to show that ζ is identically zero. Thus, in the usual manner, w and θ are the only necessary scalar quantities describing the system. To Equation (11c), then, we append

$$R^2\,\nabla_1^2\,\theta = \nabla^2\,w \quad (11d)$$

obtained by taking the vertical component of the double

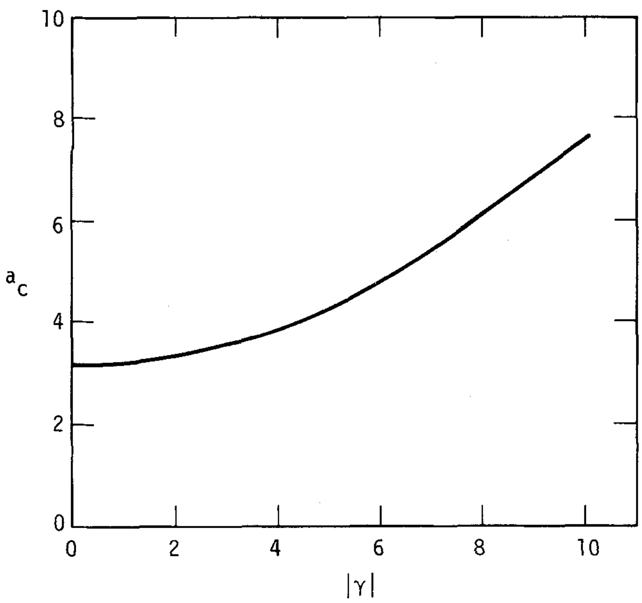

Fig. 2. The critical wave number a_c for the linear theory.

curl of Equation (11*b*) (see Chandrasekhar, 1961; and Straus, 1974).

Equations (11*c*) and (11*d*) are Fourier decomposable in the planform, yielding solutions with cellular structure characterized by a single wave number a. Thus, we arrive at the transformed equations

$$-a^2R^2\,\hat{\theta} = (D^2 - a^2)\,\hat{w} \qquad (12a)$$

$$\gamma D\hat{\theta} + \hat{w}\,D\overline{\theta} = (D^2 - a^2)\,\hat{\theta} \qquad (12b)$$

where

$$D = d/dz$$

$$\hat{\theta} = \hat{w} = 0, \quad z = 0, 1$$

The symmetry of the linear problem in the case of zero through flow may be displayed by the transformation $\hat{\theta} = \hat{\phi}/aR$. Thus, in the limit $\gamma \to 0$, $(D\overline{\theta} \to -1)$ we find

$$-aR\hat{\phi} = (D^2 - a^2)\,\hat{w} \qquad (13a)$$

$$-aR\hat{w} = (D^2 - a^2)\,\hat{\phi} \qquad (13b)$$

The set of Equation (13) is self-adjoint and admits the eigen solution $\hat{w} = \hat{\phi} = \sin(n\pi z)$, $R_n^2 = [(n\pi)^2 + a^2]^2/a^2$ with a minimum at $n = 1$, $a = \pi$, which yields the classical linear result $R_L^2 = 4\pi^2$.

For nonzero through flow, the set of Equation (12) was solved numerically for the lowest eigenvalue R^2 as a function of wave number and flow strength by using an initial value technique. Full details are available in Sherwood and Homsy (1975). The critical Rayleigh number for onset, determined as $R_L^2 = \min\limits_{a} R^2(a, \gamma)$, is displayed as a function of γ as the upper curve (dashed line) in Figure 1; the value of a for which the minimum was obtained, a_c, is shown in Figure 2.

The fact that the curve is symmetric about $\gamma = 0$ indicates that the direction of the through flow has no effect upon the numerical value of R^2. This direction does, however, have important physical significance. This is apparently the case because the base state temperature profile assumes the same shape, only reflected about the midplane, with a change in the sign of γ. For values of R^2 above the upper curve (dashed line) in Figure 1, the layer is definitely convectively unstable.

As we have mentioned, for large γ the layer becomes approximately isothermal except for a thermal layer of thickness $O(\gamma^{-1})$ near the boundary through which the mean flow exists. Thus, the fluid experiences no buoyant destabilizing force outside of this layer. In the limit of large γ then, we expect the problem to become independent of the total depth l and to scale instead with the thermal depth κ/w_o. The expectation is, therefore, that

$$\left.\begin{array}{l} R_L^2 \to R^* \,|\gamma| \\ a_c \to a^* \,|\gamma| \end{array}\right\} |\gamma| \to \infty,$$

and that behavior is in fact borne out by the numerical results. We find asymptotically that

$$R^* \approx 14.3$$

$$a^* \approx 0.759$$

These results are comparable with the values $R^* = 6.95$, $a^* = 0.429$ found by Wooding (1960) for a different boundary condition at the exit plane.

ENERGY STABILITY THEORY

The method of energy has found increased utility following its modern reformulation by Serrin (1959), Joseph (1966), and Davis and von Kerczek (1973). As we have noted, the method gives sufficient conditions for stability of the flow, without regard for the amplitude of the disturbance. It is thus complementary to linear theory in determining those regions in parameter space in which the flow is either definitely unstable (linear theory) or definitely stable (energy theory). For a concise description of the method, its objectives, and implementation, see Gumerman and Homsy (1974). The method has been applied to the problem at hand (see Wankat and Schowalter, 1970) for no mass discharge ($\gamma = 0$). In that case, the energy and linear limit are identical; thus, the condition $R^2 < 4\pi^2$ is both necessary and sufficient for stability. It will be seen below that this is a direct consequence of the fact that the D'Arcy-Boussinesq equations for $\gamma = 0$ are self-adjoint [see Equation (13)]. We now proceed to develop the predictions of the energy method for finite mass discharge.

We begin by forming the energy evolution equations in the usual manner by taking the dot product of **v** with Equation (9*b*) and θ with Equation (9*c*) and integrating over the layer, using the boundary conditions. The results are

$$\frac{dK}{dt} \equiv \frac{d}{dt}\left(\frac{<|\mathbf{v}|^2>}{2\sigma}\right) = R^2<w\theta> - <|\mathbf{v}|^2> \qquad (14a)$$

$$\frac{d\Theta}{dt} \equiv \frac{d}{dt}\left(\frac{A<\theta^2>}{2}\right) = -\quad w\theta\frac{\partial\overline{\theta}}{\partial z} \quad - <|\nabla\theta|^2> \qquad (14b)$$

where the angle brackets denote integration over the layer. Beck (1972) has incorrectly concluded that the energy method cannot be applied to this problem on the basis that for steady, inertialess D'Arcy flow, an evolution equation of the form (14*a*) does not exist. However, for this case, we apply the integral constraint obtained by setting $K \equiv 0$, and the proof goes through. We take a linear combination of Equation (14*a*) and λ^2R^2 times Equation (14*b*) to arrive at the evolution equation

$$\frac{dE}{dt} = R^2<w\theta> + \lambda^2R^2 \quad w\theta\frac{\partial\overline{\theta}}{\partial z} \quad - <|\mathbf{v}| - \lambda^2R^2|\nabla\theta|^2> \qquad (15)$$

where λ^2 is a positive constant, and $E = K + \lambda^2 R^2 \Theta$ is a positive-definite energy functional. Equation (15) may be put into symmetric form by the change of variables $\phi = \theta\lambda R$, and we have

$$\frac{dE}{dt} = R\left(\frac{<w\phi>}{\lambda} - \lambda\left\langle w\phi\frac{\partial\bar{\theta}}{\partial z}\right\rangle\right) - <|\mathbf{v}|^2 + |\nabla\phi|^2> \tag{16}$$

or

$$\frac{dE}{dt} = RI_\lambda - D$$

in an obvious notation. Now consider the maximum problem

$$\frac{1}{\rho_\lambda} = \max_h I_\lambda/D \tag{17}$$

where h is the vector space of kinematically admissible couples

$$h = \{\mathbf{v}, \phi | \nabla \bullet \mathbf{v} = 0, \mathbf{v}, \phi \; E \; C^2, \mathbf{v} = \phi = 0 \text{ on } z = 0, 1\}$$

Combining Equations (16) and (17), we find, in the usual manner

$$\frac{dE}{dt} \leq -D(1 - R/\rho_\lambda)$$

Since D is positive definite, we conclude the flow is stable for all

$$R < \tilde{\rho} \equiv \max_\lambda \rho_\lambda$$

It is also possible to prove exponential decay of disturbances in time in the usual manner, but the details are omitted (see Davis and von Kerczek, 1973).

The Euler-Lagrange equations for the problem, Equation (17), are readily found to be

$$\nabla \bullet \mathbf{v} = 0 \tag{18a}$$

$$\frac{\rho_\lambda}{2}\left(\frac{1}{\lambda} - \lambda\frac{\partial\bar{\theta}}{\partial z}\right) w + \nabla^2\phi = 0 \tag{18b}$$

$$-\nabla\pi + \frac{\rho_\lambda}{2}\left(\frac{1}{\lambda} - \lambda\frac{\partial\bar{\theta}}{\partial z}\right)\mathbf{k}\phi - \mathbf{v} = 0 \tag{18c}$$

$$w = \phi = 0, \; z = 0, 1$$

where $\pi(x, y, z)$ is the Lagrange multiplier associated with the constraint of incompressibility. It is possible by manipulations similar to those leading to Equation (11d) to eliminate π and to arrive at

$$\frac{\rho_\lambda}{2}\left[\frac{1}{\lambda} - \lambda\frac{\partial\bar{\theta}}{\partial z}\right]\nabla_1{}^2\phi - \nabla^2 w = 0 \tag{18d}$$

Fourier decomposing in the planform followed by the final change of variables $\tilde{\phi} = \phi a$ leads to the symmetric set

$$a\frac{\rho_\lambda}{2}\left[\frac{1}{\lambda} - \lambda\frac{\partial\bar{\theta}}{\partial z}\right]\tilde{\phi} + (D^2 - a^2)w = 0 \tag{19a}$$

$$a\frac{\rho_\lambda}{2}\left[\frac{1}{\lambda} - \lambda\frac{\partial\bar{\theta}}{\partial z}\right] w + (D^2 - a^2)\tilde{\phi} = 0 \tag{19b}$$

$$w = \tilde{\phi} = 0 \;\; z = 0, 1 \tag{19c}$$

From these we readily conclude that $w = \tilde{\phi}$, and for the energy method, it suffices to solve one of the (19a) (19b) for the eigenvalue $\rho_\lambda(a, \gamma)$. The optimal stability bound for the most dangerous Fourier mode is then found as

$$R_E \equiv \tilde{\rho} = \max_\lambda \min_a \rho_\lambda(a, \gamma) \tag{20}$$

We note in passing that for $\gamma = 0$, $\partial\bar{\theta}/\partial z = -1$, and it is a simple matter to show that Equation (18d) with the optimal choice $\lambda = 1$ is identical to the linear problem, Equation (13). This is a concise demonstration of the fact that $R_L = R_E = 2\pi$ for no mass discharge.

We have obtained the optimal stability boundary by solving Equations (19) and (20) by the same initial value techniques as described in Sherwood and Homsy (1975). The results appear as the lower curve (solid line) in Figure 1 and are discussed next. The energy limit at large through flow is given by the relation

$$R_E{}^2 = 5.77\,|\gamma|; \;\; |\gamma| \to \infty$$

DISCUSSION

The main results of this paper are shown in Figure 1. Given the fluid properties and mass discharge rate, the stability theory yields the temperature drop for which the layer is either definitely unstable [the upper curve (dashed line) in Figure 1] or definitely stable (the lower curve in Figure 1). Between the two limits no definitive statement can be made except to note that the layer may become unstable to finite amplitude disturbances, a so-called subcritical instability.

It is interesting to note that the curve giving the Rayleigh number for either theory is symmetric. Thus, a change of the sign of γ, that is, the direction of discharge relative to gravity, does not change the eigenvalue of either Equation (11) or (18). It is difficult to prove this symmetry property, but it appears to be due to the fact that a through flow of either sign compresses the base state gradient toward one or the other of the boundaries. The eigenvalues are sensitive only to the shape of the base state gradient, not its sense. Furthermore, it is easy to see that R_L must increase with increasing γ, since the temperature drop is felt over increasingly smaller distances while R_L is still scaled with the total depth. As Wooding (1960) has shown, and as the present calculations bear out, the stability limit must scale with the effective thermal depth as this becomes a small fraction of l.

It is important to note that even though the sign of γ does not affect the eigenvalue R_L or R_E, it does substantially affect the physical interpretation. Recall that

$$R^2 = \frac{g\alpha\Delta Tkl}{\nu\kappa} + \gamma\beta\Delta T \tag{21}$$

Thus, mass discharge in the positive vertical direction ($\gamma > 0$) increases R^2, which is a destabilizing effect, while the opposite is true for flow in the negative vertical direction. The origin of this second term in the case of mass discharge is in the temperature variation of viscosity and is therefore associated with an instability due to the motion of a stratified fluid through a porous media, and does not involve a buoyant force. As Wooding (1960) has noted, this is apparently the continuous analogue of the well-known Saffman-Taylor instability for displacement of immiscible fluids of differing viscosities. Indeed, if $\gamma > 0$ and $\beta > 0$, then the flow is from regions of low viscosity (high mobility) to regions of higher viscosity (low mobility). Such a situation in the immiscible case is known to be unstable relative to the opposite orientation. Thus, mass discharge in the positive vertical direction is destabilizing relative to the stagnant case, and discharge in the negative vertical direction is stabilizing.

We consider briefly the asymptotic situation of high discharge $|\gamma| \geqq 50$. Rewriting R^2 in terms of its definition, we find that

$$\frac{g\alpha\Delta Tkl}{\nu\kappa} = R^2 - \gamma\beta\Delta T \quad (22)$$

where $R_L^2 = 14.3\,|\gamma|$ and $R_E^2 = 5.77\,|\gamma|$.

Since $\beta\Delta T << 6$, we find in this limit

$$\frac{g\alpha\Delta Tkl}{\nu\kappa} = \left\{ \begin{matrix} 14.3 \\ 5.77 \end{matrix} \right\} |\gamma| \quad (23)$$

Introducing the definition of γ and rearranging, we are left with a very simple stability criterion involving a dimensionless group we choose to denote as the D'Arcy buoyancy number

$$B = \frac{g\alpha\Delta Tk}{\nu_o|w_o|} \quad (24)$$

For $B \leqq 5.77$, the layer is unconditionally stable; for $B \geqq 14.3$, it is certainly convectively unstable. Since $|\gamma|$ is large in many practical situations, these simple asymptotic criteria are often applicable.

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NOTATION

A = ratio of volumetric heat capacities
a = wave number
B = D'Arcy buoyancy number
c_p = heat capacity, J/kg · K
D = differentiation operator, d/dz
E = energy functional
g = acceleration of gravity, m/s^2
h = vector space
I_λ = energy production integral
K = kinetic energy
k = permeability, m^2
k_m = effective thermal conductivity, J/m · s · K
$\mathbf{k}$ = unit vector in the vertical direction
l = layer depth, m
p = fluid pressure, Pa
R^2 = D'Arcy-Rayleigh number
T = temperature, K
ΔT = imposed temperature difference, K
t = time, s
$\mathbf{v}$ = velocity vector, m/s
w = vertical velocity
w_o = through-flow velocity, m/s
z = vertical coordinate, m

Greek Letters

α = coefficient of thermal expansion, K^{-1}
β = coefficient of viscosity variation, K^{-1}
γ = dimensionless through flow, w_ol/κ
ϵ = porosity
ζ = vertical vorticity
Θ = norm of disturbance temperature
θ = dimensionless temperature
κ = thermal diffusivity, m^2/s
λ = coupling constant
μ = viscosity, Pa · s
ν = kinematic viscosity, m^2/s
Π = Lagrange multiplier
ρ = density, kg/m^3
ρ_λ = eigenvalue defined by (17)
$\tilde{\rho}$ = optimum eigenvalue
σ = D'Arcy-Prandtl number
ϕ = scaled temperature field

Subscripts

0 = reference conditions
f = fluid property
m = mixture property
L = linear theory prediction
E = energy theory prediction

Superscripts

$\wedge$ = Fourier transformed variable
— = base state variable
$'$ = disturbance variable

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